JOURNAL OF MATHEMATICAL BEHAVIOR **10**, 117–160 (1991)

# Inventing Graphing: Meta-Representational Expertise in Children

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We examine a cooperative activity of a sixth-grade class. The activity took place over 5 days and focused on inventing adequate static representations of motion. In generating, critiquing, and refining numerous representations, we find indications of strong meta-representational competence. In addition to conceptual and design competence, we focus on the structure of activities and find in them an intricate blend of (1) the children's conceptual and interactional skills, (2) their interest in, and sense of ownership over, the inventions, and (3) the teacher's initiation and organization of activities, which is delicately balanced with her letting the activities evolve according to student-set directions.

## 1. INTRODUCTION

In November 1989, 8 sixth-grade students in a school in Oakland, California invented graphing as a means of representing motion.

Now, of course, we mean that they "reinvented" graphing. In fact, we know that most of them already knew at least something about graphing. But the more we look at the data, the more we are convinced that these children did genuine and important creative work and that their accomplishment warrants study as an exceptional example of student-directed learning. We would like to understand

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We thank the following for comments on drafts and for contributing to the ongoing discussion of inventing graphing: Steve Adams, Don Ploger, Susan Newman, Jack Smith. We are immensely grateful to the 8 sixth-grade students who did the creative work presented here. This project was supported by the National Science Foundation. The opinions expressed here, however, are those of the authors and do not necessarily reflect those of the Foundation.

This research is supported primarily by a grant from the National Science Foundation, NSF–MDR–88–50363, to Andrea A. diSessa. We also gratefully acknowledge a contribution to supports students from Apple Corporation, and a contribution of equipment from Sun Microsystems.

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how inventing graphing happened, in the hope of arranging similar events in this and other contexts. Thus, the goals of this work are:

1. *To describe what happened.*
2. *To describe what we believe students exercised and learned.* We are concerned, in particular, with meta-representational competence, by which we mean the faculty to generate, critique, and refine representational forms. Here, this means designing paper-and-pencil representations of motion. We use "meta" to describe these capabilities to emphasize that no specific representational skills are implicated. Unlike schooled or any automated capabilities students may have for using established representations, the skills we attend to are broadly applicable, more flexible, and fluid.
3. *To begin to analyze how the activity worked.* In this we will focus, in particular, on the structure of the activity. We intend to look at broad factors in the flow of the discourse as well as some of the moment-by-moment interactions of teacher strategies with student characteristics. We want to understand the *activity as set* by the teacher, the *activity as interpreted* by the students, and the *activity as negotiated* on an ongoing basis.

## 2. ORIENTATION

### 2.1 A Child's Science of Motion

We begin by setting the context for the work described in this article. The largest subproject of the Boxer Project at Berkeley is one we call "a child's science of motion." The goal of this, and essentially, of all our current work, is to demonstrate compelling new models of learning in cultures supported by a substantially altered representational infrastructure. Simply put, if pencil and paper, books, and concrete materials are extended by a computational medium such as Boxer, we believe more powerful modes of learning will become accessible.

A child's science of motion is guided by two key principles. First is a principle of continuity of ideas: We wish to discover and build on substantial expertise that children already possess. This is a deeply constructivist orientation that knowledge flows from prior knowledge.[1]

The second principle of our work derives from a less familiar orientation. The orientation is that, with regard to learning, the structure of activity deserves equal concern compared to our concern with knowledge. We do not pretend to espouse any general theory of action and knowledge, but we maintain that the former is important to the acquisition of the latter, if these can be separated at all. Knowl-

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[1] Constructivism, instigated largely by Piaget and his colleagues and followers, is, of course, a broad and influential point of view in contemporary studies of learning. A reference that outlines some of our own theoretical principles and instructional design oriented around constructivism is diSessa (1986).

edge flows from and emerges in activity, so the design of activity should be the first priority of whose who would aid learners. Indeed, the competence and appropriate willingness to engage in a certain class of activities constitute important educational targets in their own right.

Coherent activity of any sort, certainly coherent intellectual activity, is dependent on many things. It depends on knowledge (in a very general sense) or competence. But it also depends on personal goals and interests, not to mention habits, attitudes toward learning, and attitudes toward the particular subject matter. In class, it also depends on interpersonal goals and means, social structure, and interactional skills.

The concern for activity as a focus then leads us to a second principle of continuity. Not only must we base our instruction on a thorough understanding of children's ideas and how they relate to our target scientific concepts, we must understand the possibilities inherent in children's capabilities for coherent action. To design instruction means to chart a path from "childish" activities toward more scientific ones. It emphatically does not mean to teach children to emulate the overt activities of scientists, "experiments" or "the scientific method," any more than it is appropriate to think of children as appropriating scientific knowledge on a blank slate.

## 2.2 Practical Orientation

We are exploring these principles in the design of a course on motion for children. We chose to work with upper elementary-school children because we believe that they have a pool of intuitive knowledge which is quite sufficient to support sophisticated understandings of motion. Yet, there is no shortage of documentation that motion is difficult to learn by conventional means.[2] In addition, motion meshes well with the special characteristics of a computational environment. For example, computers are excellent at dynamic graphics. From the perspective of activity, we believed that design, construction, and exploration of dynamic games and simulations would provide a rich context for an initial exploration into what children's science might involve. Accordingly, we chose to take our root metaphor to be the child as engineer—as builder—rather than the child as (scaled-down adult) scientist. More generally, we wanted to explore widely to avoid prejudice as to what activities can be both child-appropriate and, in a genetic sense, genuinely scientific.

The course is designed to occupy a full year. Time is a key parameter in designing for cultural and deep conceptual change. Our curriculum is ambitious.

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[2] See, for example, Viennot (1979), McCloskey, Caramazza, and Green (1980), Trowbridge and McDermott (1980, 1981), Clement (1982), and McDermott (1984). A review of the literature on learning graphing that includes a discussion of many known difficulties, and misconceptions is Leinhardt, Zaslavsky, and Stein (1990).

It covers the core concepts of kinematics usually taught in high school or university physics, including:

Time, distance, and speed
- Velocity and acceleration
- Qualitative versions of differentiation and integration

Graphing
- Generation and interpretation of position, velocity, and acceleration graphs
- Translation among the different types of graphs, position, velocity, acceleration

Vectors
- graphical addition
- two dimensional velocity and acceleration

Composition of motions
- Relative motion
- Frames of reference

Instructional modes in the course are diverse. We have designed a number of microworlds focusing on some key concepts or phenomena. Careful analysis of the engagement and learning of children in these environments is a major part of our work. Although many of the microworlds involve specific, prescribed activities, we designed each microworld to open into more student-driven activities and projects. Much of this openness is a direct result of the use of Boxer.

In addition, we are working to develop a supportive environment for student programming projects in order to allow a more personal orientation toward learning, and to provide for deeper contact with some motion ideas than uniform curricula permit. Finally, we chose to focus particularly on group discussions as an interesting class of activity structures for learning. This focus is inspired in part by recent excellent examples of classroom discussions as learning activities, such as the "benchmark experiences" of Minstrell (1989).

The particular episode in question, inventing graphing, falls into the last category, discussions. It illustrates many of our central goals, strategies, and concerns, as we will make evident. Yet, it has little directly to do with technology. To be sure, work with computers fed into this event, and much computer work followed upon it. But the event itself consisted essentially of 5 days of discussion, 30–40 minutes per day, among the children and their teacher. We are happy to illustrate in the analysis of this event that technology, although a central tool in our overall plan, does not indiscriminately dominate our concern for children's learning. Other articles will more than adequately show the indispensability of the medium to our overall means and success.

## 3. SETTING

### 3.1 Students, Class, and Teacher

The class consisted of 8 bright and generally articulate, but not inordinately precocious, sixth-grade students in an academically oriented private school in Oakland, California. Aside from balancing gender, we made no particular attempt to select a representative population. Given the challenging curriculum and project goals, we did not feel an "easy start" would jeopardize our results. In addition, the orientation of the project is toward establishing and studying models of success, not yet toward dissemination and direct assessment of generalizability. The student names we use are fictitious, but the sexes indicated are correct.

The class was taught by Tina Kolpakowski (Ms. K), who has an undergraduate degree in cognitive psychology and 5 years of teaching experience at middle-school levels. She specialized as a teacher in mathematics and studied introductory physics, but she had never before taught physics in any form. Her experience with computers involved some programming in several languages, teaching elementary Logo programming, and using Boxer informally in some of her classes in 1988–1989. She collaborated intensively with the rest of the Boxer group in the design of the course, which was accomplished partly in the preceding summer, and partly "online" during the school year.

The motion course was an elective for the students. It met as the last class in the afternoon, nominally 40 minutes per day, 4 days per week. There was no assigned homework. Although we originally organized 1 afternoon per week for after-school computer time, we added additional sessions at the students' request. Those were devoted exclusively to children's independent projects. Students' attitudes toward electives at the school are generally playful, without very serious commitment to achievement.

All of the students in the motion course were also in one of Ms. K's (as the students called her) sixth-grade math classes. Ms. K often had students in math discuss their ideas in a roundtable format, taking turns explaining their own work, attending to, and critiquing—respectfully!—the explanations of others. The students had enough practice with this type of discussion that, when Ms. K asked them to explain their ideas to each other "like in math class," they all understood what she meant.

Inventing graphing took place 6 weeks after the beginning of the course. The course began with 2 weeks of instruction in programming. The students then spent 2 weeks programming computer simulations of various "real-life" motions. These simulations included the motion of a book shoved across a desk and a ball rolled off the edge of a table. After that, they spent a week trying to determine and simulate characteristic speeds, such as the typical speed of a car and of a person walking. This involved conversions between different units of speed as well. In the days before the inventing-graphing discussions, they programmed simulations of a car driving at varying speeds. We will describe this later as it pertains directly to the discussions.

### 3.2 Set Task

Among the Boxer group, the activity proposed to the children was coded "inventing graphing," though we had no idea how accurate that label would be. (The children knew it as "motion pictures.") In fact, we thought of the activity as a bit of a wild idea, meant mostly to set get some data on spontaneous representations used by children (inspired, e.g., by the work of Bamberger, 1989, on children's representations of rhythm), and to set a meaningful context in which "proper" graphing could be introduced. The teacher, in particular, hoped only that this activity would provide a context in which the students would immediately understand and appreciate the value of graphing when it was introduced in a more traditional fashion.

The initial description of the activity was simple. The children had been making simulations of motion. Now we were giving them a harder job. They were to design a static motion picture, as expressive as possible, but within the constraints provided by a piece of paper. That is, there could be no real motion. In addition, the teacher emphasized that it should be as simple and easy to interpret as possible.

### 3.3 Data

The first of the five class sessions was audiotaped, the rest were videotaped. In addition, at least one graduate student observer was present at most classes and wrote field notes. These were especially helpful for the session that was not videotaped. All five sessions were transcribed.

## 4. ANALYTIC FRAMEWORK

Our presentation of data will be interpretive, but in a middle ground. We do not intend to direct our interpretations solely toward proving some particular points. Nor will we be unselective, presenting "everything that happened." Rather, the presentation will be organized around a number of themes, with heightened consideration of events that relate to them. But we also present at least minimal consideration of events we take to be worthy of mention, whether or not they relate directly to principle themes. In this way, we hope to present a balanced view of what happened without being reticent about what we take to have been important about it. There are two main lines we wish to follow in our exposition, one conceptual and the other focusing on the structure of activities.

## 4.1 Conceptual Line of Analysis: Meta-Representational Competence

The first priority of investigation is the range and adequacy of resources that children can bring to bear on the problem of representing motion. This should be especially evident in the range of representational forms that they invent. Equally important is their capability to critique given representations. Here, the range of criteria that they can bring effectively to bear is foremost. Responding appropriately to critiques brings to light further inventive resources. In addition, we would like to know how articulately aware these children are of representational resources, constraints, and other such issues. In addition to the broad categories of invention and critique, a number of subthemes of conceptual development will emerge.

***Discrete to Continuous Models.*** The class started with discrete representations of motion in segments of constant speed. Over the course of the discussions, these evolved to continuous representations. We find the transition interesting, with possible implications regarding meta-representational criteria as well as underlying conceptualizations of motion.[3]

***Figural Influence.*** Representations of motion will have their own global figure or gestalt. One would like to think of scientific representations as reflecting only properties of that which is represented. But there is evidence children attend to figural features of the forms, sometimes, but not always, to the detriment of strictly representational functionality.

***Time Versus Distance.*** Conventional graphing of motion usually means explicitly representing time. (We note, though, that this is hardly necessary. It is often useful to represent velocity as a function of position, such as describing the motion of a fluid.) The route the students took to time-based representations was not straightforward. They spent a good deal of time discussing whether to represent time, distance, or both, with regard to general criteria like representational parsimony and clarity, as well as with regard to specific questions like how to show the duration of a stop.

***Representational Competence.*** At a more general level, we are interested in whether this meta-representational approach supported competence with representations. Were students able to use each other's representations? Were there indications that their understanding of graphing was substantively improved?

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[3] Graphing continuous motions discretely is a "misconception" noted, for example, in McDermott, Rosenquist, and van Zee (1987).

***The Move to Appropriate Abstraction.*** The students had programmed a simulation to represent a car moving through the desert. Now we were asking them to attend only to particular aspects of the story, to the motion of the car defined in scientific terms, abstracted from the situation. We are interested in how their understanding of the task developed to this level of description.

We must further be aware that the nature of conceptions of motion must be implicated in attempts to represent it. By and large, these children showed adequate control over most of the issues raised in inventing graphing. Speed seemed to be quite salient and easy to reason about. They seemed to have adequate understanding of, for example, the relations among duration, speed, and distance covered. On the other hand, some of the later discussions indicate difficulties with the concepts of signed speed and instantaneous stop.

## 4.2 Pragmatic Line of Analysis: The Structure of Activity

We will refer to issues concerning the structure of activity as "pragmatic." Because of the way the teacher ran the task, these centered largely on the dynamics of discussion: Who has control? Who can speak and for how long? How is the topic selected and agreed upon? Whose ideas are picked up? How diverse or homogeneous is the opinion of the participants? How do they balance their personal lines of thinking with the group dynamic? How flexible or rigid is the flow of the discussion, bounded by the teacher's setting and the agreed topic?

Perhaps the most fundamental issue for us is how the pragmatic line articulates with the conceptual. Are central conceptual issues raised explicitly, implicitly, or suppressed? Is adequate time and focus built into the activity in order to capitalize on and build conceptual/design insights effectively? Do conceptual and pragmatic goals align sufficiently so that both may be satisfied? More particular pragmatic considerations follow.

***Ownership.*** Ownership of ideas and artifacts is a potential advantage to having students design representations. Did these children own and feel that they owned the ideas developed? At a finer scale, how was ownership shared in the group? Did individuals hang onto their own creations, adopt the group consensus, or adopt the ideas they perceived to be best, independent of originator and independent of the feelings of the rest of the group?

***Interest.*** It surprised us how much interest was exhibited by the group, although this varied among the participants. But there is much beyond this that one would like to know about the focus and kind of interest expressed. How did interest affect, and how was it affected by, the dynamics of the discussion? To what extent did it originate in the content of the discussion, and to what extent in the social dynamic?

***"Revisiting."*** Conceptually difficult topics or unusual design moves may simply require more time than may be naturally allocated by pragmatic considerations. Which topics or ideas were revisited, and how did that happen?

***Teacher's Stance and Strategies.*** Many classrooms are dominated by teachers' agendas and judgments to an extreme degree. This was not the case here. However, the teacher made many strategic moves and decisions that bear especially on the pragmatic line. Many of these concerned issues already mentioned: allowing and fostering student ownership, arranging interesting frameworks for action, and prompting revisitations of crucial issues.

In summary, the analysis that follows is focused primarily on tracking the evolution of representations of motion. When possible, we comment both on inventive and critical capability. We seek to display accomplishments or difficulties related to conceptualizing motion. And finally, we wish to gain some insight into the structure of activities, to understand the character of the activity as it played itself out and how pragmatic considerations articulated with conceptual ones. Our coverage is broad because our program is to uncover children's expertise that may be of use in other learning designs, as well as to discover what made this one work.

## 5. ANALYSIS

Days 1 and 3 are the most critical. We encourage readers to study both the overview and more detailed exposition of these. Days 2 and 4 are less critical; a reading of the overviews should be sufficient to make our synthesis in Section 6 meaningful. The preliminary days, which took place before what we consider to be Day 1 of the episode, and Day 5 are presented in sketch only. In addition to our interpretations, we interpolate some representative selections from the transcript. Figure 1 provides an overview and brief synopsis of what happened to help readers orient themselves.

### 5.1 The Preliminary Days: Introducing the Task

In the 3 days before the inventing-graphing discussion, the students worked on creating a simulation of what we called "the desert motion":

> A motorist is speeding across the desert, and he's very thirsty. When he sees a cactus, he stops short to get a drink from it. Then he gets back in his car and drives slowly away.

They worked on this in pairs for 2 class periods. All the groups produced simulations in which the motion of the graphical object, a Logo-like turtle, involved segments of motion at constant speed. This is not surprising because we provided example simulations that had this same form. In addition, the students

## Preliminary Days

- The task is introduced, and Ms. K asks for five words that describe the desert motion.
- Julie draws the first motion picture.


The first motion picture.

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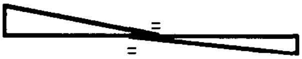
## Day 1

- A large collection of representations are invented.


"Triangles"


"Slants"


"Chalk"


"Eiffel"


"Sonar"


"T's"


"Dots"

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## Day 2

- Ms. K. asks for more words that describe motion.
- The previous day's inventions are restated and clarified.

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## Day 3

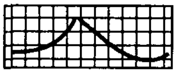
- Mitchel has an "awesome" idea: slants hooked end to end.


Connected Slants

- Mitchel draws a continuously varying version of the desert motion using his new idea.


Continuous Version

- Steve suggests "grids" and graphing appears. They will name this new representation Niagara.


"Niagara"

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## Day 4

- Representations are named.
- They play a game of depicting a given motion using a variety of representations.
- There is a discussion of whether a graph should resemble a hill which appears in the description of a particular motion.

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## Day 5

- There is a discussion of whether a stop occurs in a motion which involves a reversal of direction.

**Figure 1.** An overview of inventing graphing

programmed their simulations so that the turtle left a trail of dots across the screen, spaced farther apart the faster the turtle moved.

This representation—dots spaced farther apart to indicate speed—was a starting point for the invention of other representations. It had an interesting prior development, which is worth taking a moment to discuss. In the earliest simulations of an object falling, only one group, Charlie and Mitchel, thought to show the object speeding up. At first they made their simulation with dots coming closer together toward the bottom of the fall. Questioned by an observer about the fact that the turtle actually moved more slowly toward the bottom of the fall, they asserted they meant only to depict more speed (evidently shown as "more happening," or else as a greater density of "events," like running footsteps). They did not mean to have the turtle actually move faster. This was a very early indication of how students would sometimes approach the task of representing motion abstractly. With the prompt from the observer that they might make the simulation more realistic than symbolic, they quickly fixed the program so that the turtle actually sped up, the dots now getting farther apart as it fell. When the rest of the class saw this in a group discussion, it met with significant surprise but relatively quick approval. The dot representation became standard in the class's computer work.

Julie and Amy were the first group to finish programming their desert motion simulation, 2 days before the discussions began. The teacher reminded them about the motion-picture part of the assignment, which she first described as a "picture of how the car moved." Their first reaction, like that of others later, was to point at their screen showing the desert road, including the car and cactus, and the trail of dots after their simulation had run. In order to give them a better understanding of the task, Ms. K asked them to describe how the car moved using only five words. She dismissed words like "desert" or "cactus" as having nothing to do with the motion, and eventually Julie came up with "fast, abrupt stop, slow, fast."

Mrs. K then pressed them to come up with a way to "draw a picture that shows those five words." Still they had trouble understanding, but, without other instruction, Julie was able to create the first motion picture (see Figure 2). It had a solid horizontal line to indicate fast motion, a space marked off by vertical lines to indicate the stop, and dots to indicate slow motion.

The next day, the rest of the students first began the motion-picture task. Bobby and Charlie, like Julie and Amy, initially tried to present the final state of their simulation as a solution to the task. They then amended their simulation, adding a vertical line to address the teacher's objection that the picture on the

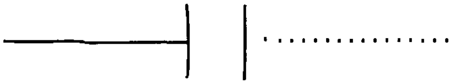

**Figure 2.** Julie's first motion picture (our rendition)

screen did not show how long the car was stopped. Mitchel and Karen, who did not get to the task until the end of the period, only got as far as claiming that their simulation screen sufficed. Ms. K presented the five-word task to Steve and Sharon, who spent awhile trying to understand what kinds of words were appropriate, with the teacher asking them, for example, whether "drinks" tells anything about how the car moved.

That afternoon, the Boxer group discussed the progress students were making, and we decided we should turn the motion-picture task into a class discussion away from the computers. We now begin the description of the discussions that followed.

## 5.2 Day 1: A Feast of Ideas

***Overview.*** On the first day, Ms. K reminded everyone what the task was about, emphasizing that there were two parts: generating a picture, and explaining it clearly to others. Students began working in pairs on their pictures. While they did this, Ms. K responded to questions about what was proper and what was not allowed in the drawings. She deflected most questions; she did discourage the use of words in the representations, though not the idea of a "key" to the picture. After about 10 minutes of work the class reassembled. Ms. K had the students present their ideas in a roundtable, "like in math class." (The students were familiar with this format of discussion, as previously mentioned.) Each student presented her or his ideas in turn, with other students asking questions for clarification or giving polite critiques.

The discussion was lively: both critical and approving. Generally, the students genuinely seemed to wish to understand each other's ideas. From time to time, Ms. K focused on crucial questions such as: Which motion picture is simplest? Which shows the stop? Which shows the duration of the stop? Children continued to improve or add to their own representations as they heard the ideas of others. Ms. K prompted frequently for reasons for redesign, or reasons for why one design was preferred to another.

***Detail.*** The presentation here is in rough order of public introduction of the representational forms. Several of the representations underwent improvements during the discussion, however, so it is not easy to pinpoint their genesis. The names of the representations listed here were mostly not assigned by the class until Day 4. We use them here for convenience.

*Dots (Figure 3).* This was the familiar representation from the students' computer work, dubbed "dots" or "spot" by the students. It is not surprising that

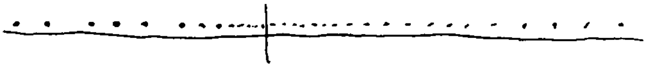

**Figure 3.** Bobby's version of Dots

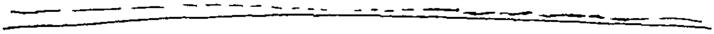

**Figure 4.** Karen's version of Chalk

it appeared several times during Day 1. Figure 3 shows Bobby's version. Like almost all of the early representations, it includes a horizontal line intended, at first, as a picture of the road. (Julie's representation, described before, is a notable exception.) Eventually, the students came to think of the line as a representation of distance, or of some ambiguous measure of duration, rather than as showing the road. This allowed them later to consider explicitly whether it would be better to use the horizontal axis to represent time or distance.

*Chalk (Figure 4).* This representation, introduced first by Amy and then again by Karen, seemed to be a variation on dots. As described by Amy: "Ok, this is my picture. See, the longer the line is, the faster it's going. When it's just a dot, that means it's stopped." The metaphor of chalk was introduced by another student, Charlie, who brought up Amy's idea when it was his turn to speak. He said he liked her idea because it showed what would happen if someone in a car were dropping chalk dust, or lifting and putting down a piece of chalk, at regular intervals. Generally, the students did not take care to assure spaces represented equal time intervals, but merely served the need to break up the line. The students discussed this point:

Charlie: What are the spaces in between the lines?
Amy: Well, how can I interpret how long the line is without spaces?

Despite the strength of the concrete metaphor, the length of the line was consistently described as representing speed, not a distance. This is a small indication of the directness and accessibility of speed for these students.

*Triangles (Figure 5).* This was Steve's idea, and it was usually labeled in the class with his initials. He was representing the motion he had simulated, which was of the car overshooting the cactus, stopping briefly, and then backing up to it, where it stopped for a longer time. He indicated speed by the vertical spacing between the horizontal and slanted lines, and the stopped states with equals signs, with the longer stop represented by the larger equals sign. Steve's horizontal line represented distance along the road—or the road itself—not time, so he needed to show the duration of the stop with another method. The reason for the second slanted line's appearance below the horizontal was not clear, and seemed

**Figure 5.** Steve's Triangles

**Figure 6.** Eiffel (Ms. K's rendition for the summary sheet)

either aesthetic (Steve: "It wouldn't look right") or to avoid overlap with the upper slanted line. In later presentations, Steve drew both triangles above the distance axis.

Triangles were difficult for the others to understand, in part because of its author's description of the "thickness of the lines" (rather than distance between lines) as representing speed. Under their questioning, he did seem to get the basic idea across. Part of his response was to color in the two triangles (as shown in Figure 5), presumably to make them seem more like thick lines.

*Eiffel (Figure 6).* This representation was called Eiffel because, in one of its renditions, it looked like a tower turned on its side. There, the speed was depicted by the thickness of a horizontal line, similar to the convention used by Steve in Triangles. Versions of this were produced by Bobby, Charlie, and Julie. Charlie included a vertical line to indicate the stop. Eiffel did not generate much discussion, but it seemed to be the basis for several variations.

*Sonar (Figure 7).* Sonar, by Charlie, is notable for using the vertical dimension cleanly to represent speed. It was one of the few early representations that did not show a horizontal line to indicate the road. In fact, the horizontal dimension in Sonar, as in some other representations, ambiguously showed time or distance. Note, in particular, the short horizontal segment indicating the stop.

The distance–time ambiguity may have been exacerbated by use of discrete representations (some versions of Sonar and Eiffel; Slants and Ts, in the following), in which sequence is sufficient, and more salient, than any global metric meaning along the horizontal. Getting the question of time versus distance along the horizontal axis cleared up was one of the major successes of the inventing-graphing discussion. But it was a difficulty that also seemed to persist later in the course as students inadvertently slipped back and forth between use of horizontal for time or distance, starting soon after inventing graphing finished and persisting to the end of the course.

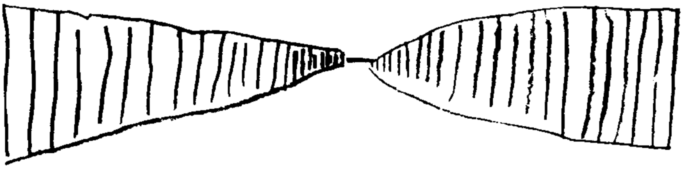

**Figure 7.** Charlie's Sonar; later renditions did not have the envelope.

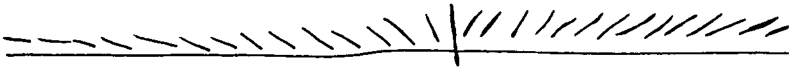

**Figure 8.** Mitchel's Slants

*Slants (Figure 8).* This was a substantial move that may have been of particular importance in the path toward standard graphing. Mitchel explained that his key idea was to use the slant of the segment to represent speed: "If the line is horizontal, he's going really really fast. And the further up the line slants, the slower it goes. And then when it gets like this (vertical), it is a stop." He described a horizontal line on several occasions to represent "as fast as the car can go." Note that this intuitive landmark, "as fast as possible," provides a natural end point to the scale, but such naturalness is quite unnatural from a scientist's perspective. It is too arbitrary and related to the particular object involved in the motion.

Although slants might reasonably be conjectured to have developed from "speedometer sampling," we do not believe this was the case. First, the "vertical is stopped" convention is wrong for speedometers. Second, Mitchel was quite articulate about using the dimension of slope to represent speed, both in his initial presentation (just paraphrased) and in recaps during later days. So, even if he were adapting a real-world model, he was very conscious about its abstract status as using one continuous dimension (slope) to represent another (speed). Last, the slants representation used positive or negative slope equivalently, which is entirely inconsistent with speedometers.

This point, that slanting right indicated the same thing as slanting left, was made by Charlie, though it did not seem to be meant or taken as a criticism. Charlie said, "You could show it either way." From Mitchel's explanations, it appeared that the gestalt of the representation, gestured as a continuously turning and sweeping pencil, might well be responsible for the lack of distinction between positive and negative slopes. The continuous sweep was simply too compellingly nice to ignore. On several other occasions students made judgments on the basis of such figural aesthetics, sometimes to the detriment of the integrity of the representation. The author of slants, in particular, seemed more and more to be taken by the appearance of continuity of various sorts in representations, one example of which will be reported later.

Slants was generally well received, although it was criticized as being confusing for someone coming from outside the class. Amy argued, "Well, my only problem is that . . . if someone was to look at it, they'd look at it and go, 'Ok, what does that mean?' " Mitchel responded, saying that it needed the same amount of explanation, "one sentence," as other representations. Mitchel generally supported and developed this representation, but he also actively engaged in developing other ideas as well, at several points professing to prefer them to slants.

*Ts (Figure 9).* The T representation played an important role, both pragmatically and conceptually, in the evolution of the discussion. In it, the horizontal lines came to represent speed, apparently following chalk, and the vertical lines came to represent time. Note that in this form, time is doubly represented: once in the length of the vertical bars, and once in the horizontal sequence. This did not appear to bother any of the students.

The T representation was pragmatically pivotal in raising some key issues. It was the context in which the relations of time, speed, and distance were first exercised and discussed at great length with respect to representational implications. It became well recognized, for example, that the product of the horizontal and vertical lengths would give the distance traveled in each little segment of motion. The fact that any two values would determine the third was frequently discussed, and came to prominence, for example, as an argument against one student who proposed to add another horizontal line below each T "top" to represent the third quantity. The economy of using only two determiners quickly won this battle with respect to Ts, but the issue of whether it would be good to show all three if possible persisted in several discussions.

The genesis of the T representation, which we describe in the following paragraphs, was especially interesting as very many important ideas emerged in this somewhat surprising pragmatic context. Note how much comes from an attempt to redesign a fairly innocuous defect in a prior representation. Only a small part of this was specifically "nudged" by the teacher.

After all the students had a chance to explain their work, Ms. K opened the floor to general discussion and criticism of all the representations. At one point the discussion turned, at the instigation of Ms. K, but with easy compliance by the students, to how well the various representations showed the stop and the duration of the stop. Triangles, one of the few to represent the duration of the stop, drew criticism for using an ad hoc symbol, the equals signs, rather than something more closely related to the rest of the representation. Amy complained, "Well, you wouldn't look at it and go: 'Oh, the equal sign means he stopped.' " Mitchel questioned why one would use two symbols (lines and equals) instead of one. He, in fact, on another occasion, showed great pride in the fact that the stopped state in slants is just a natural continuation of the representation of slower and slower movement: "I think mine is kind of realistic because it relates to the rest. Like slowing down has to do with the stopping

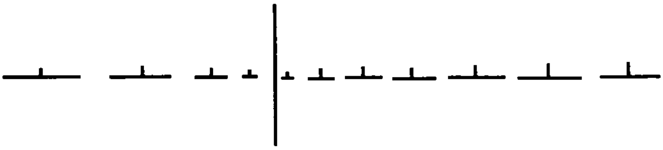

**Figure 9.** Ts (Ms. K's rendition for the summary sheet)

because it gets more and more this way (vertical) as it slows down, and all the sudden, it stops."

From a request by Ms. K to make suggestions on how to improve on the use of equals signs, Steve, the author of triangles, after muddled attempts by others, proposed the use of numbers for speed. When Mitchel commented that this use of numbers was not helping with the question of duration of stop, Steve chimed in with "a number every 5 seconds." Thus proposed, was a representation of motion as a list of numbers, which was further elaborated by Mitchel and Bobby. Mitchel explained:

> Well, we could do something like without a road at all, just like a computer readout, where it's the number of numbers tells not only how fast it's going, but how long it stays going that speed. So you could go like . . . 10, 10, 10, 9, 9, 8, 7 as it slows down, and then the number of zeros shows how long it stops. . . .

Note the close analogy to chalk: a speed sampled every few seconds. In fact, the interval suggested here, 5 seconds, was the same as that discussed with chalk, and chalk's presentation would shortly be augmented with a vertical line to produce Ts. The students elaborated lists of numbers as a representational form. Steve suggested alternating numbers for speed and numbers for distance in the list. This was quickly argued away as too complex (and, presumably, redundant).

Ms. K praised lists of numbers as "getting very accurate." Indeed, the students voiced strong support for them as the currently preferred way to represent motion. However, Ms. K eventually asked if there were some way to accomplish the same ends without numbers. Mitchel noted that "we have two dimensions, one can mean one thing and another can mean another," and he proceeded to draw and explain the inverted T representation, speed on the horizontal, time ("how long it stays at each speed") on the vertical. He noted how cleanly Ts handled the duration-of-stop problem. He praised Ts, saying that they "would be even more accurate than numbers and take less space." The day's class ended here.

It is worth noting that several times in the course of the discussion, students criticized representations not only for lack of clarity and such, but for inaccurately portraying the given motion. For example, one showed speeding up after the stop at the same rate as the slowing down, whereas the desert motion had the driver leaving in a leisurely manner after coming quickly to a halt. Each such remark was an occasion to exercise motion picture reading capability, which gradually evolved into graph reading and generating capability. In a standard class on graphing, this might be the only focus, accuracy of representation. In this class it was largely ancillary to design.

## Some Hypothetical Genetic Observations

The central task of this article, to uncover child competence, is a tricky one. It is tricky because the point of uncovering competence is to build on it, and the ways competence can develop are not always evident in the competence itself. Thus, use of discovered competence may frequently be hypothetical, looking toward future possibilities not evidently realized.

In this context, we consider a few *hypothetical genetic* possibilities.[4] These are conjectures on how some of the ideas that came from these children might be involved in building more expertise. Hypothetical genetic considerations are not tentative claims for inevitable routes to learning. They identify resources, and sometimes, particular ways to use those resources in promoting the development of scientific ideas.

The first example from Day 1 is a simple one. Sonar shows, in a perceptually salient way, a feature that is buried in standard graphing: the use of the vertical dimension to represent speed. Thus, this invention might attune students to the particular representational resources used in graphing.[5]

A more complex example, and one that probably would require more instructional attention, comes from slants. Slants, like Sonar, provides an excellent feature decomposition (using discreteness), and it provides a preview of the notion of speed represented as the slope of a distance/time graph. To be clear on these relations, imagine cutting a distance versus time graph into smallish, discrete segments, and bringing them each down to center on the time axis. Except for a reversed convention (by usual conventions, vertical means infinite speed, not stop), this is the slants representation. Alternatively, hooking slants' segments end to end produces (qualitatively, and with the convention reversal noted above) a time versus distance graph. This hooking together, in fact, was later proposed by a student as an improvement to slants, adding to the case for slants as a potentially very productive contribution. We mean in this to imply that working with slants in various forms can help illuminate less explicit use of slope in graphing instruction and reduce the appearance of "misconceptions" such as slope–height confusion (Clement, 1989; McDermott *et al.*, 1987; van Zee & McDermott, 1987).

Ts, with time and speed on the two dimensions, could be a step toward an inscribed-rectangle method of integration (Riemann sums). The students recognized that the product of the two dimensions would give the distance travelled in each little segment of motion. Rather than push the T representation toward rectangles, one might annotate a graph of speed versus time with "hidden Ts" to show children an approximation to integration, based on their own observations (Figure 10).

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[4] "Genetic," of course, means developmental in the most general sense. DiSessa (1982) and Kliman (1987) discussed such possibilities in the context of "learning paths" that represent both observed and hypothetical child developments integrated into a perhaps multipathed "curriculum."

[5] Clement (1989) noted a similar point. He also made the historical point that Oresme (and, we note, Galileo) used such "skeletal" annotations in their proto-graphs, and that there is at least anecdotal evidence that 10-year-olds can readily produce speed versus distance graphs in this form.

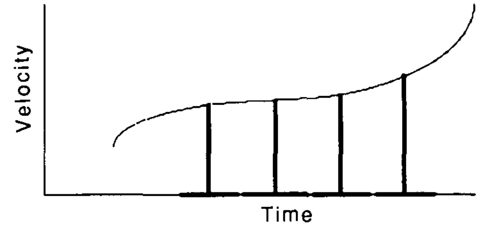

**Figure 10.** A T annotation of a velocity versus time graph: "Integration" as the children understood it.

Lists of numbers, which emerged in the development of Ts, had a less-than-hypothetical genetic future. We instructed this representation much later in the course. Teaching it was not a result of its spontaneous development; we had planned to use it all along. But its spontaneous appearance at this early date helped validate it for us as an easily appropriable representation. When we taught number lists, we used separate distance and speed lists, in part to emphasize the idea of them as unified entities (functions).

## 5.3 Day 2: Abstraction and Consolidation

***Overview.*** The second day was a full-class discussion. No really new representations emerged. The evident work done was to clarify the major classes of representation under consideration. Ms. K also intended to develop a better sense of the range of "scientific motion phenomena" as a sharper target by which to judge representational expressiveness and clarity. Thus, kinematics deals only with spatial amounts and rates as opposed to dealing with causality, intentions, object constitution, or any other aspects of a motion's context.

The class began with some students providing, at Ms. K's request, a summary of what had happened the previous day for a student who had been absent. This was also presumably to refresh everyone's memory (Day 2 was a Monday and Day 1 had been the previous Thursday). The summary described the task, to invent a motion picture and explain it, and described the final representation, Ts. Ms. K then spent about 10 minutes eliciting a long list of words that could be used to describe kinds of motion. This was a deliberate extension of the five-word description task previously described. She explicitly noted her intent to use the list to challenge representations.

The class then started with the T representation and the desert motion again. The announced, but never achieved, plan was to vary features of the motion to test the representation. The students did substantial work clarifying the meaning of the T representation and discussing variations on it. One by one and on their own initiative, students added all the major classes of representation previously discussed, and there was a critique of each exemplar's faithfulness to the desert motion, for example, did it show an abrupt stop and slower acceleration away from the stop?

***Some Details.*** The student chosen to draw the first T representation, Bobby, added a key: "horizontal lines = speed" and "vertical lines = distance." Although others did not notice immediately, there was a growing confusion about what the dimensions actually did or should represent. Eventually, Charlie and Mitchel asserted that the vertical *should* represent time, which Ms. K sanctioned. She then asked students to explain each T of the representation, what it showed, and whether it was faithful to the desert motion. During this, Mitchel remarked intently that this was "more like a motion chart than a motion picture."

Mitchel: This is looking more like a motion chart than a motion picture.
Ms. K: Ok, why . . . ?
Mitchel: Because it's getting—this is like a readout you'd get from a computer than something that you can just draw. . . . There's no distance. It doesn't show any distance or, you know, the amount of space. It's just a couple of aspects.

We took him to be suggesting that they were producing a technically precise and abstract depiction rather than a more realistic picture. He made this comment in similar forms on several occasions, and not only on this day.

Another student (Julie) offered a "totally different but the same" picture, which turned out to be a neater and more elaborated T picture, with more Ts and with the vertical line crossing the horizontal one symmetrically.

Charlie introduced the possibility that the horizontal line could represent distance, leading to a new version of Ts. He drew this on the board, and, incidental to his purpose, happened to make all the time intervals look equal. Ms. K asked if there were any advantage to that, but no conclusion emerged. His version was discussed, with no definite conclusion. The advantages of the T representation in general, compared to chalk, were articulated mainly as representing two aspects of the motion, not just one.

Ms. K: What I want to know is why did we change from the chalk idea to this horizontal and vertical idea? Were there advantages to it or—
Mitchel: It shows only one aspect and the vertical (T representation) shows two.
Bobby: (If) you use the chalk idea, it always is going for a 5-second interval. But if you use the other one—
Steve: This (one T) could be going for 5 seconds here and 10 seconds here. . . .

Mitchel reintroduced slants, this time with consistent use of positive slope. One student (Charlie) asked about the motion "between the lines."

Charlie: My only problem with that is what's between the lines?
Mitchel: What's between what lines?
Julie: Yeah, 'cause yours' shows speed, but it doesn't show time.

In Day 3, this issue of continuity would be resolved.

One by one, the other representations from Day 1 were offered, mostly by their authors, with minor inventive variations to make clearer one point or another about the desert motion. There were occasional references to computer possibilities during this day, which were mostly suppressed or ignored by Ms. K. Bobby suggested he could say how to program Ts in Boxer. Late in the class, the idea of having speed numbers pop up in the display of the motion picture was introduced. The class closed with Ms. K saying that their job for the next day was to find a way to combine all the best features of these pictures, and to find a way to show numbers without actually writing numerals. Thus, she was pressing on the criterion of getting precise numerical measures into the picture.

Though nothing dramatic happened on Day 2, the activities seemed important for making sure the class was together and for moving toward more stringent criteria for adequate motion pictures. A number of mistakes and confusions indicated such needs. Among our general concerns, the following were touched: the distance versus time issue, continuity, students' perceptions of the task of representation ("picture drawing" versus a technical, selective representation), students' capabilities to critique and compare representations, and one or two interesting teacher's moves.

## 5.4 Day 3: Breakthrough: Continuity and Grids

***Overview.*** Day 3 was an extraordinarily active, collaborative, and productive session. It began, innocuously enough, with Ms. K handing out a set of drawings she had prepared showing all of day 1's representations. She set the task of naming each picture, but this activity got quickly derailed into a discussion of whether it would be desirable to show all three aspects, time, distance, and speed, or only two. This was instigated by a dispute about whether the T picture should show distance or speed on the horizontal. Several students interrupted each other to explain the central point that any two of the three would do. In this discussion, Mitchel volunteered that he had an "awesome idea" that combined everything into one. His representation was basically slants, hooked end to end. The length of each segment was to represent distance, slope represented speed, and a possible (but not necessary) transversal on each segment (à la T) showed times.

Steve proposed adding a grid to this representation, which now had a very graphlike appearance, so one could easily read off distances on the horizontal and speeds on the vertical. (The fact that the grid and the hooked-together slants were incompatible was not noticed, or it was not articulated by anyone with the possible exception of Mitchel.) Grids were greeted with considerable enthusiasm: "That would be cool." After a somewhat chaotic period of clarification and modification, Ms. K asked the students each to make a drawing of their favorite motion picture, using whatever ideas they thought best. Four of the six drawings were unambiguously graphs of velocity versus distance; one was a graph of velocity versus time; one gridded graph was unlabeled; and one student preferred the discrete version of slants, using length to represent distance and transversals to indicate time.

***Detail.*** During the beginning discussion of names for pictures, it became clear that there was a dispute about whether the T representation would be better with distance or speed on the horizontal. Amy complained that, obviously, the issue was speed, so presumably, distance was not appropriate. A somewhat chaotic discussion followed, with several students interrupting each other to explain matters to Amy. This was punctuated by Amy's complaints that her remarks were always "snapped at." The conceptual point of the discussion was that knowing any two of speed, distance, and time, the third could be determined. From things the children said in this context, and things they did on the computer, it seemed to us that all of them had this idea under control. In this context, Bobby and Sharon offered quite clear explications. (Sharon: "The up-and-down line is the time it took it to go the side-to-side distance. If the line gets longer going up and down, that means it took a longer and longer time to go that distance.") Nonetheless, the discussion continued along these lines. At this point, Mitchel offered his "awesome" idea, explaining that it was a combination of other ideas.

Mitchel took the floor by moving to the chalkboard to illustrate as he talked. He began by explaining that a vertical line could be maintained from Ts to represent time, although several times he subsequently approved of the economy of representing just two aspects. Then he recapped the key idea of slants: Slope represented speed. He emphasized that the length of the line was irrelevant to its slope and then commented that one could use the length to indicate distance. Finally, he commented that one needn't separate line segments as before, and drew a zigzag line (Figure 11), narrating the meaning of the lengths and slopes.

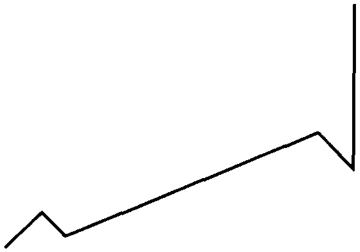

**Figure 11.** Mitchel's connected slants (our rendition)

Note how he plays on the sign ambiguity of slope to articulate segments of the motion, harking back to more discrete representational forms.

Charlie pressed the point that time was not necessary and that distance and speed were enough. Mitchel accepted this point.

| | |
|---|---|
| Charlie: | Right, but so you wouldn't need time. |
| Mitchel: | The time would still be there. |
| Charlie: | Why would you need it? |
| Julie: | How? (Meaning, how would the time still be there?) |
| Mitchel: | I mean, you wouldn't really. |
| Charlie: | I know, so you wouldn't need it, so why are you saying you need the time? |
| Ms. K: | You wouldn't even need time? |
| Mitchel: | You wouldn't even because it's the speed and the distance which makes up the time. . . . |

Ms. K asked, apparently warily, if this was a completely different motion picture. Julie took a lull in the conversation to suggest another modified T, with a third line, drawn dotted and horizontally, representing distance so that all three aspects were depicted. This met resistance; two aspects were held to be sufficient. Ms. K suggested that Mitchel show the desert motion with connected slants so the class could better understand his new idea. She said she was also uncertain of its meaning.

It is important to note the quality of the conversation, especially here and in Day 5. Topics were introduced, seemingly dropped or combined, later to reemerge effortlessly, continuing lines long dormant. We believe it is important to allow students to pursue slightly independent lines, maintain their individual engagement, and be part of the group at the same time. As on other occasions, the size of the class is implicated in the pragmatic line meshing appropriately with conceptual issues; in a group design such as this, ownership would be lost if too many students' ideas were to get lost or unheard.

Just as Mitchel was about to begin using his idea to show the desert motion, Steve complained that, for showing stop, the vertical line should be "nothing because there's no distance." He was apparently reacting to the long vertical line at the end of Mitchel's zigzag slants, which, according to Mitchell, represented a long stop. At first, Mitchel resisted, declaring that he needed some length to show a slope.

| | |
|---|---|
| Steve: | Mitchel, won't your stop be nothing because there's no distance? |
| Mitchel: | There's got to be a little line, because otherwise you wouldn't be able to represent it (slope). But it would still . . . It would just be a dot! |

Mitchel retrenched by suggesting that "we could just use a space" to represent the stopped motion, which is what he went on to do.

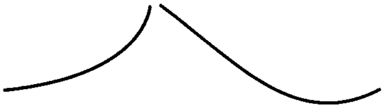

**Figure 12.** Mitchel's depiction of the desert motion using connected slants (our rendition)

Mitchel began drawing the desert motion while Ms. K insisted he explain what he was doing as he went along. The figure he drew was roughly as in Figure 12. Note that the slope is now continuously varying. He explained the blank space as depicting the stopped state, "there's nothing and nothing . . . there's no distance and no speed." Using absence to represent zero seems primal, though, as with the number system, it needs to be transcended.

Charlie, once again, noted the ambiguity in Mitchel's use of positive and negative slope. He drew an alternate version of the second half of the graph flipped about the horizontal to illustrate (Figure 13). Mitchel again accepted this possibility, but explained it as a global symmetry rather than a local one. He said Charlie's graph was just his, rotated by 90°. He illustrated by showing his graph as a half circle, facing downward, with a small gap taken out (he literally erased a small segment from an approximate semicircle), and Charlie's as a similar shape, oriented as a normal C.[6] He assumed a global rotational symmetry whereas the symmetry has really only to do with part of a graph, and it is a reflection, not a rotation.

Two things are interesting about this exchange. First, Charlie seems to have Mitchel's representation under better control than Mitchel with regard to symmetry. This was not at all unusual in the course of the discussions. Students frequently chimed in with cogent critiques, noting other students' inconsistencies with their stated intentions and descriptions, and the like. Note, for example, that the small intervention of Steve, just described, prepared Mitchel for further necessary moves in improving his representation. In terms of the cognitive dynamic, this role of commenting students is quite comprehensible. Individuals have to focus their attention on some aspect or other of their activity at the expense of others. But the rest of the class, not having primary responsibility for that particular action, or having their own expertise to add, help build a more coherent and cogent overall product. It is our guess that the discussions would not have worked with a much smaller class because of the critical mass of ideas generated by this group, and because of the scaffolding provided by others in the discussion. It seems unlikely that a teacher alone could provide such sufficient scaffolding, particularly because her contributions have a quite different status from those of the students, but also because she has so many other managerial roles to respond to as well.

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[6] In glossing Charlie's graph as a C, Mitchel has inadvertently combined a piece of Charlie's and a piece of his graph.

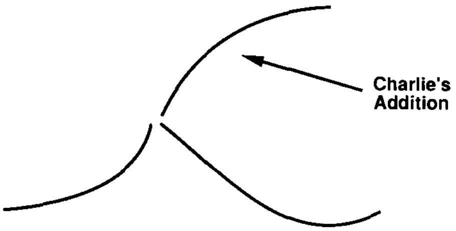

**Figure 13.** Charlie's modification of Figure 12 (our rendition)

Second, note that Mitchel, in explaining the symmetry relation of his and Charlie's version of the graph, glosses both graphs in a simple gesture. We noted before that such figural considerations were fairly common. They provided independent considerations that motivate aspects of a depiction. As with Mitchel's sweeping pencil image that completed for him the second half of slants, figural considerations may not always be optimal. Here, however, the simplified figural version of the graph seems to be at the root of Mitchel's recognition of symmetry as the core issue.

During much of the preceding and following, one of the students (Amy) complained frequently that this all was too complicated, that if people hadn't been in the Boxer class, they'd never understand any of these pictures. "We've been in Boxer class for like a few months, so we had time to learn and understand about this, and these people are only gonna be there for a few minutes." At this point Ms. K responded to the complaint by saying, roughly, that students had an idea, and now they were in the process of simplifying it. She directly reminded students that the final product of an idea doesn't always resemble its introduction.

After Mitchel and Charlie had drawn the curve shown in Figure 13, Steve joined in to say, "put a grid on it." His suggestion was greeted very positively. Ms. K sent him to the board, and what followed was a stunningly collaborative creation. First, Steve cleared away much of what was on the board, but retained Mitchel's connected-slants depiction of the desert motion. Then he drew axes and labelled them, with zero speed being at the top, where Mitchel had depicted the stopped state. As Steve continued, Bobby came spontaneously to the board to correct the second half of the drawing, explaining that it was going too low, which would mean the car drove away faster than it arrived. Mitchel suggested that the conventions be changed so that the scale would have the bigger numbers at the top. He seemed to be suggesting, incorrectly, that this could be accomplished by reversing the slants' conventions connecting slope and speed. Steve interrupted him to say "you could flip it." Mitchel agreed that this was what he was saying, adding, "You slow DOWN and speed UP."

Apparently to bring some order to the chaos, Ms. K asked each of the students to draw their own favorite idea, first individually, and then on the blackboard.

Grids dominated. Most of the students used rulers at their seats and then at the board. All but one of the drawings were on grids, the one exception being slants augmented with a transversal to represent time. One of the grid drawings was unlabeled; four had speed versus distance labels, and one was labelled speed versus time. Ms. K especially complimented Karen's for being neat and simple, but later (implicitly) criticized it for not having any labels.

To conclude the class, Ms. K recounted similarities and differences in the drawings. She asked a series of questions, such as "whose shows the stop; whose shows it driving away slower?" The students' responses were all appropriate. For example, the speed versus time graph was criticized as incorrectly leaving out the duration of stop.

We conclude our own discussion of Day 3 by remarking that hooked-together slants and Ts could be incorporated neatly in a graph of *position* versus time. In Figure 14, the angles are modified Ts, with horizontal length representing time and vertical representing distance. In this form, the segment of graph across each L is a slants segment, again with reversed convention: Horizontal is stopped, vertical is fast (infinite speed). Aside from adjustments to cohere with accepted conventions, the students in Ms. K's class provided representations that could be used as insightful annotations of a standard position versus time graph: They are perfectly valid, serviceable representational forms.

### 5.5 Day 4: Tuning: Pictures Versus Graphs

***Overview.*** Ms. K announced a game. The students would be given a motion to depict in the different representations, and then they would compare the pictures to see which portrayed that motion best. It seemed evident on a couple of occasions that Ms. K favored conventional graphing, but she made no open advocacy.

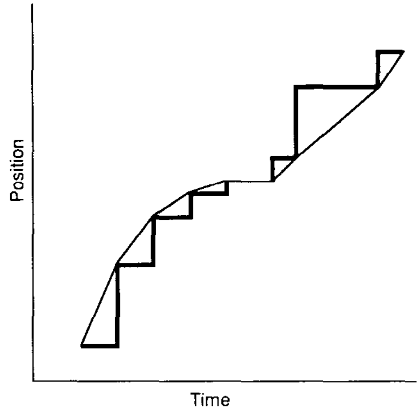

**Figure 14.** Position versus time graph incorporating Ts (modified to Ls) and Slants to highlight slope

As a preparation to the game, the students finished the aborted task of deciding names for all the representations, including the new ones introduced the previous day. Ms. K indicated that they should resolve the conflict of whether the horizontal axis should represent time or distance before continuing. Although only one student had advocated time at the end of the previous day, the discussion quickly resolved in favor of time on the basis of its representation of stop. During the presentation of simple motions, all the representations produced were reasonably accurate. Students suggested improvements, such as adding a scale, to forms that seemed initially at a disadvantage in capturing certain aspects of motion. Graphing generally seemed to have the favor of the group, but not decisively so.

At one point, a student drew a graph of a motion up and down a hill (slow at the top) with the speed scale inverted, so that the graph looked like a hill. Animated discussion of the disadvantages of this revolved around conventions (what people will expect) and how abstract a representation this is intended to be. The conclusion was in favor of the abstract, graphical representation. The class ended with an apparent consensus that graphing speed versus time is the best, simplest representation.

***Some Details.*** Ms. K handed out drawings she had made of the prior representations, including graphing and the augmented-slants representation (length of the slant indicating distance, and a transversal representing time). Graphing was named Niagara because in it the desert motion looks like a falls, and the augmented slants were called "Telephone Poles." The variation of graphing with an inverted-speed scale was called "Volcano." On some occasions, students used Volcano to denote a graph with distance and not time represented on the horizontal.

Ms. K suggested that the group should settle the conflict of using time or distance on the horizontal axis of graphing. Charlie, who had championed time the previous day, announced his continued advocacy of it. In the ensuing discussion, several students participated and gradually clarified the issue as relating to depicting the stopped state. Ms. K scaffolded the discussion and ended it by focusing on the central point. To leave a space, which they all wished to have in order to show a duration of rest, one must use time. A speed versus position graph cannot have any segments along zero speed because it is nonsensical to move a distance at zero speed. Agreement on this argument seemed solid. A segment of the discussion helps catch its tenor and the role of the teacher in helping with a difficult point.

Ms. K: . . . we really need to find out what that bottom thing is supposed to be. What were the choices?
Bobby: Time and distance.
Ms. K: Bobby.

Bobby: It's either distance or time.
Ms. K: Either distance or time.
Bobby: If you have it time, then you can't have the space.
Charlie: What do you mean? Yes you can. You can have the space. You took—you went no speed, which is this no speed right—for that amount of time.
Steve: If it was *distance*, you can't have the space.

Steve: It would be going a distance at zero speed. You can't do that.
Ms. K: You would be going a distance at zero speed. Oooh. Does that make sense to you?
Sharon: Yeah.
Charlie: And that's the problem with distance, and that's why I think we should use time.

The game of depicting different motions with the various representations began with a constant speed motion, described by the term for constant speed the students had agreed on in the words discussion on Day 2, "cruise control." The motion was introduced with a story about someone driving on the freeway at 50 mph. The students were enthusiastic and suggested a competitive game, boys against girls. Ms. K refused and explained that this is a game everybody plays just for the fun of it.

Four students drew Sonar, Ts, Dots, and Niagara on the board, all roughly correctly. Sharon, who had drawn Niagara, added a scale to it to indicate the 50 mph precisely. Steve, who had just drawn Sonar, drew the motion in Triangles as well. Charlie interrupted the game saying that he was thinking that, with the T representation, they could *choose* to show *either* speed or distance along the horizontal, depending on which they were interested in. Realizing that the choice of representation depends on its use is one of the major meta-representational advances one would hope for, and this is one child's small illumination in this direction. Bobby commented that Niagara is much better with a scale. Although Niagara seemed to be ahead in popularity, students added scales to Sonar and Ts, which improved them. Ms. K asked which representations could not depict a certain speed easily, and Steve answered Triangles, which had been his invention, and Dots.

The second challenge in the game had a woman biking up a hill, pausing at the top to rest and then riding down. The point was eventually made that she goes much faster downhill than uphill. Among the representations on the board was Niagara. Julie had drawn it roughly correctly, a U-shaped graph with a gap at the bottom indicating the rest stop, where she added a picture of a cactus. In the course of discussion, Bobby criticized the graph because "I don't see it going up the hill." Although he apparently meant to say that the graph was symmetrical and didn't show a slower speed going up than going down, his remark was interpreted as a criticism that the depiction was too abstract. Julie, who, as with her cactus here, had added roads and other naturalistic features to her depictions on occasions, inverted both the scale and the graph so that it looked like a hill. We note that the realistic depiction of topographical and other features in place of graphed quantities is a commonly cited graphical "misconception." Here, the misconception is benign, because it involved only a nonstandard convention. But it provided another context in which to discuss the purpose of graphing and some potential difficulties to avoid.

Criticism of the hill depiction was very quick. Steve noted that people would look at the shape first, not the scale, and be confused: "That's kind of hard. Most people look at the graph before they look at the scale. They think, 'Oh, I see, it's going faster and going slower.' " Most appeared to favor the conventional scale, but a few seem to favor realism. Julie even went so far as to add a hill parallel and underneath the graph, supporting the cactus, which she kept in her "graph." Steve continued to complain that it was a picture, not a graph. The discussion ended as Ms. K pushed for representing the attribute of speed, not other features like hills or cacti. The class ended with Ms. K asking for the students' favorite motion picture. Some began to propose their own creations. Ms. K asked them to try to put aside personal favoritism and think which depiction was best. There seemed, then, to be firm consensus that Niagara, graphing speed versus time, was best.

We note that ownership and pride in production seemed quite evident on this day, as on others. For example, Julie, as she had done on the previous day, added her name to her drawing. In fact, as the class ended, she walked directly past the camera and asked it, "Isn't my graph nice?" But intellectual ownership of the core-developing ideas was looser. Essentially, all the students changed their advocacy from one representation to another relatively fluidly. All attended enough to others' depictions to critique or suggest improvements in them, or to absorb ideas for later, or for their own creations.

### 5.6 Day 5: Does It Stop, or Not?

Day 5 began with Ms. K continuing to ask children to depict various motions in different representations. The first motion was that of a bicycle rider who does not have the strength to get to the top of a hill. When he "runs out of energy" going up a hill, he cleverly is able to ride his bike backwards down the hill. Ms. K's focus was on graphing a reversing motion and negative speed, but the group never really reached these issues. Instead, a dramatic and extended dispute arose over whether this person stopped at the moment his motion reversed.

The debate is interesting for a number of reasons. First, it exhibited a number of features about the students' models of motion. In particular, it showed that continuity through zero speed was not a salient property. Initially, most of the class seemed to support the contention that the rider need not have stopped between moving forward and backward. The discussion also showed that many of the students held a model of "stopped" that necessarily involved an extended duration; instantaneous stopping was, for them, a contradiction in terms. The properties of the debate were interesting in terms of activity structures as well.

> The debate shows what kind of intense involvement was possible for these children in conceptual issues concerning motion. The teacher only barely managed to suppress discussion at the end of the period, when she took over the class to explain what she intended for them to do at the following meeting. Just as the bell rang, a student began again to make a point about the stopped state, and the children left the class continuing to discuss the issue.
>
> The debate involved more serious and central participation on the part of the girls in the class than had been the case in many instances previously.
>
> At one point, Amy complained that Ms. K should just give them the answer. Ms. K responded appropriately, we think, that she was not sure there was an unequivocal "right answer," and, besides, she felt they could figure this out if they tried. This was the only clear instance in the entire discussion when these children asked Ms. K to assert authority.
>
> Finally, Ms. K totally abandoned her original agenda to follow the debate. We believe this is indicative of a commitment to children's ideas and their intellectual ownership of the class.

On the other hand, there were only a few issues in the discussion that related to graphing. Because of this, and because the analysis of this class deserves more extended discussion than we can afford here, the rest of what we present is only a sketch of the issues that arose in Day 5, with particular attention to those related to graphing.

Before the debate began, and as students were putting different depictions of the hill scenario on the board, a trio of students debated privately what was apparently implied by one of the student's graphs, that one could move backwards in distance. Charlie and Sharon considered this absurd. But Steve, who had drawn the graph in question, maintained it made perfect sense. Steve was, in fact, the only child who gave any indication of appreciating the meaning of negative speed at this point, and, perhaps not coincidentally, he was the probably the staunchest supporter of the existence of a stopped state in the hill motion. Thus, it is plausible that he maintained there must have been a stopped state by virtue of continuity between positive and negative speed (though his explicit arguments did not involve this point).

Steve had drawn what he called Volcano, by which he presumably meant it had distance on the horizontal axis. His plot was shaped like a backwards C, with arrows along it to show in which direction the curve was to be read. Julie used Niagara, drawing a curve that swooped down toward zero speed, without touching the axis, and then reversed itself to return to a high, positive speed. The question arose as to whether the graph should touch the axis, should show zero speed, and hence the debate began. During its course, no one questioned whether, in fact, the graph should include negative speeds.

There were two other relatively interesting issues in the debate concerning graphing. First, at one point Ms. K introduced the idea of greatly expanding the graph to make a point about small time intervals. The thought experiment was accepted as relevant and, in the end, taken to be convincing by the student against whose position it was raised. The other point was subtle. During one period of the debate, the question arose as to whether it would be reasonable to make a graph that might be technically correct if it would be also misleading to graph readers. There was no clear resolution to this.

The inventing-graphing discussion was terminated at the end of Day 5, with some grumbling from the students. Although it was evident to us that more time could have been productively spent exercising representational skills and motion conceptions in this context, the tyranny of schedule took us back onto the planned track.

## 6. SYNTHESIS

We return to our opening questions as a way of summarizing, abstracting, and synthesizing our account of inventing graphing. What happened? What did the students exercise and learn? How is it that the activity worked? Afterward, we take the perspective of instructional design to draw out some implications.

### 6.1 What Happened?

We summarize briefly what we consider evident and impressive.

> *The students invented, critiqued, improved, applied, and moved fluidly among a diverse collection of representational forms.*

We believe that it is not a bad first approximation to say that these students (re)invented graphing, even though we know that most of them already knew something about graphing. Graphing had not been part of their sixth-grade mathematics class, but on a pretest given before the start of the course, two questions involved choosing the correct speed versus time graph for a given motion; 5 students answered both correctly. However, the long developmental path exposed in this article suggests there is really a lot more to learn about graphing than they already knew.

Of much greater interest to us—as well as more directly evident in the data—is that the class showed substantial meta-representational expertise. They invented, evaluated, and refined a variety of representational forms. Along the way, they introduced or successfully applied a number of criteria concerning the quality of representations (of course, they did not use this language):

- *Transparency.* A representation should need little explanation.
- *Homogeneity.* Use the same notation for a stop as for motion, not equals signs.
- *Compactness.* Ts take less space than number lists.
- *Conceptual clarity.* Ts are "precise," showing the requisite two aspects cleanly.
- *Objectivity.* "Could be done by a computer."
- *Appropriate abstractness.* Show only aspects; needn't show the road, cactus, and so on.
- *Faithfulness.* Continuity in speed is better expressed in continuous graphs.
- *Completeness.* Can derive all three relevant aspects of motion: speed, distance, and time; can show all kinds of motion, such as stop.
- *Economy.* All three aspects of motion can be derived from any two that are presented.
- *Quantitative precision.* This was introduced by the teacher, but adopted easily by the students.
- *Consistency.* Conventions should not be adjusted for particular motions, such as for the bike on the hill.

Moreover, the students showed the ability to interpret and generate representations of specific motions within the forms they invented. This was especially evident on Day 1, when criticisms often focused on the accuracy of the depiction rather than on the invented mode, as well as later, on Day 4, in the game of representing particular motions.

> *The students engaged in a cooperative design activity, partially organized by the teacher, but substantially run, especially at the scale of introducing and developing ideas, by the students.*

Our epistemological stance goes beyond a concern for knowledge in the traditional sense. We believe that what happened pragmatically deserves as much attention as what happened conceptually.

The activity was very interesting and engaging for the students. Indeed, the teacher spent far more time constraining and channeling the conversation than she did trying to get it going. Almost all of the substance of the discussion came from the students, including all the representational forms and many of the criteria for judgment. To be sure, the teacher's role was important in *organizing, focusing, and characterizing,* but it was much less important in terms of bringing ideas to the table, and even in judging them.

The students listened to each other's ideas seriously and with interest, asking questions to clarify, and offering suggestions (e.g., Steve's first presentation of Triangles was unclear, so others pressed him to explain; Charlie pointed out Mitchel's ambiguous use of slope for slants). More impressive, they used and elaborated on each other's ideas, and they were often quite explicit about it (e.g., Charlie provided the metaphor of chalk for Amy's dashes; almost everyone contributed in some way to elaborating Steve's suggestion of using a grid). Their contributions were certainly not equal, but at some point almost every child contributed substantially to the learning that was going on.

### 6.2 What Did They Learn?

It is difficult for us to measure exactly how much the students learned for two reasons. First, these students were operating as competent designers, both conceptually and pragmatically. To be sure, this is consistent with our orientation toward thinking of children as having substantial resources, and of instruction as often accessing and developing capabilities more than supplying them. However, in such cases what is new to students may not be so apparent as when competence follows incompetence.

Second, we have more information about the students as a group than we have about them as individuals. Although the group used and developed various competences, the extent to which each student "had them" is not clear. This is especially true of their meta-representational skills. We think it is beyond question that none of the students had the capability to invent and critique representations in anything like the richness that was exhibited by the group. On the other hand, it is important information that the group displayed a competence. There are two ways to read this. First, we can take a Vygotskian perspective and think of this group as working in a zone of mutual proximal development.[7] Individuals may be internalizing (or reflecting on) what is first produced in a group.[8] A more general social-constructivist view is that the group might also have been developing irreducibly social competences, abilities that function only in group contexts and are dependent on group properties. Each of these views captures part of what we believe happened.

With these caveats in mind, and interpreting the term broadly, we describe what we believe students learned.

*The students developed their understanding of the construction and interpretation of speed versus time graphs. More important, they did this in a properly meta-representational context in which the purposes of graphing and the general repre-*

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[7] See articles by Cole (1985), Wertsch and Stone (1985), and Brown and Ferarra (1985) for a discussion of internalization and the zone of proximal development. Wertsch (1985) is a good general reference on Vygotsky.

[8] Our data challenges narrow readings of Vygotsky in that we don't believe it is possible to localize the support being provided in an "instructor" or in more competent student colleagues.

*sentational criteria they satisfy are salient, and in which graphing is seen as one option among many.*

The ability to construct and interpret speed versus time graphs is, by traditional curricular standards, the most obvious target of learning here. Although students knew something about graphs before the lesson, we claim they sharpened that knowledge considerably. They practiced drawing and reading graphs. Students made mistakes and displayed imprecisions that other students noticed. They explicitly confronted cases we know to be problematic for students learning to graph: whether the graph of the speed of a bicyclist on a hill should resemble a hill; how to show backwards motion.

We claim also that because these students learned at a meta-representational level, they were moving toward a richer and more flexible sense of the use of graphing to represent motion than is typical in instruction. They learned functions of graphing, that it is for the purpose of communicating and thinking about motion. They learned that it has various advantages and disadvantages compared to other representations, and compared to what we might like a representation to do, such as to show transparently all possible features of that which is represented. They not only learned why graphing is good compared to some other representations, but they also developed some alternatives that might be more valuable on certain occasions. In the course of the discussions, they had at least some practice with lists of numbers, graphs of speed versus distance, and bar graphs (Sonar, in contrast to line graphs).

They learned about representations as a class of conceptual objects. They showed a solid understanding of the difference between a representation and the thing being represented, noticing, for example, that a representation may show "only a couple of aspects," and had no difficulty with the task of applying a variety of representations to a particular motion. They talked about the difference between a technical representation like graphing and less technical ones, like "drawing a picture."

In designing, they thought about representational resources generally (e.g., the suggestion of using two available dimensions to depict two aspects of motion), and also more specifically (e.g., either slant or thickness can be used as a representational dimension). They applied all the criteria we listed for evaluating their representations, explicitly discussing many of the criteria.

*The students learned about motion.*

If they did not know it before, certainly they learned that two of the three aspects of uniform motion (speed, distance, time) are sufficient to determine the third. We believe, as well, that the students were exercising a piecewise-constant intermediate mental model of motion. Especially early on, and manifest in the T representation, they thought of motion as a sequence of durations each characterized by a speed. We believe this is a very productive stepping stone to competence with continuously changing motion, and elsewhere in the course, we explicitly taught it.

The students exercised the notion of continuity of speeds in the question of what was "in between" the pieces of a discrete graph. The idea that zero is just one possible value for speed was implicit in the discussion of the merits of using the same kind of symbol to show a stop as to show motion. As simple as it seems, it is easy to underestimate the difficulties involved in assimilating absence or stop to the same numerical quantification as other "amounts." Consider these students' use of gaps to represent no motion.

They learned something of the richness and subtlety of other motion concepts through an attempt to represent them. For example, they gained better control over "speeding up" (acceleration) versus "speeding up faster" (increasing accelerating), and they discussed problematic aspects of an instantaneous stop even if they did not fully resolve and consolidate the issues on Day 5.

*The students sharpened their capabilities to act as a community of designers and inquirers.*

We mentioned early in this article that the competence and willingness to engage in a class of activities should be considered educational targets in their own right. In this class, the students practiced a mode of collaboration in which they attended to each other's ideas, criticized them, appropriated, and elaborated them. We will have more to say about this later.

### What Was Not Learned?

In addition to accomplishments, it is as important to discuss limits that appeared in learning. These alert us to problems that may need attention as well as calibrate our expectations for success. We will treat only limits in motion- and graphing-related concepts. Meta-representation is new enough as a focus that we do not believe it is profitable to speculate on a targetable range, and how far short these students might have come.

We already noted that negative speed appeared to be problematic. Its lack of consideration seems to emerge from the feeling that displacement, distance, is always a positive quantity. Some students explicitly rejected the concept of negative displacement. On the other hand, we have very little calibration on how difficult a hurdle this is. The instantaneously stopped state is clearly a problem that needs further consideration.

Although it seemed that the notion that time is a better horizontal axis choice than distance was agreed to, we noted that students early on were quite ambiguous about which one they were using. More telling, even after the inventing-graphing discussions, we noted instances where students seemed not to attend to the distinction in drawing their graphs.

### 6.3 How Did It Happen?

That this activity worked as it did was a great surprise to us. We had originally planned only a day or so of exploring ideas. The success of Day 1 had us scrambling to decide whether to let it continue and to videotape the discussions. The question we pose for ourselves here is: How could these discussions happen?

To organize our discussion, we present a crude model of the instructional task from our standpoint as designers. The central focus for design, as we proposed earlier, is the structure of activities. We consider the design of activities from the perspective of the two orienting principles:

> CONTINUITY OF IDEAS: Instruction must apply and build on conceptual expertise children already have.
>
> CONTINUITY OF ACTIVITY STRUCTURES: Instruction must apply and build on pragmatic expertise children already have, that is, on their capability for, and interest in, pursuing particular kinds of coherent activity.

Thus, we will seek to lay out prior conceptual and pragmatic expertise of children as it contributed to these activities. We will focus on student interest as a significant factor. Finally, we will also single out the teacher's role for attention. In principle, a teacher can make conceptual contributions to an activity, and she can make pragmatic ones.

Perhaps the main goal of activity design, after capitalizing on these continuities, is to provide for proper articulation of them. We want the activity, as designed, to provide a *frame for meaningful action,* making use of and extending students' prior pragmatic skills at the same time that those activities engage their level of conceptual attainment.

Roughly, then, we see inventing graphing to have been the frame for meaningful action within which conceptual issues, representational and meta-representational issues, and issues relating to motion were elaborated. Child inventions were not the substance that we wanted them to learn, but the concrete focus of the activity from which knowledge was developed.

***Conceptual Expertise.*** **We believe the most significant result of this analysis is the discovery of substantial meta-representational expertise in children.** These students knew a great deal about what good representations are; they could critique and refine them. They showed substantial inventive capability, dramatically more than what we had imagined. From our point of view, that they had such expertise was essential to their interest in and engagement with the activity, as we will elaborate in the following.

We do not know of any prior published description of such knowledge in children as it relates to scientific representations. Obviously, what we provide here is only a very preliminary analysis, based on one case. It raises questions, however, regarding what we generally assume children are capable of doing, questions that merit further attention.

Their expertise regarding motion contributed as well. It is important, though easy to miss, that students found speed easy and obvious as a primary quantity, rather than as a rate of change of a different primary quantity, distance. This fact seemed to show itself continually. On the graph-reading part of the pretest, the students did much better on speed than they did on distance problems. After inventing graphing, students had much more difficulty graphing distance than speed, and on the final exam, several students graphed speed when they were requested to graph distance. This was true despite the fact that we tried systematically to exercise both distance and speed versus time graphs. Thus, we think student focus on speed manifests that speed is a principal and direct resource in students' thinking about motion.

***Pragmatic Expertise.*** Similarly, the success of these discussions must have relied on the students' substantial interactional expertise. Students were not only generally attentive to each other, but were capable, even when on uncertain intellectual ground, of entering into conversations with critics and emerging with changed and improved points of view. Consider Mitchel's encounters on Day 3 with criticism from Charlie and Steve in which he stood his ground reasonably, but emerged with better ideas concerning how his slants could or could not represent stop, and how appropriate it was to display all three aspects of motion.

To be sure, these students were somewhat special. They came from academically oriented families; they were enrolled in a similarly oriented school. However, it is too easy to attribute their interactional skills to such factors. Academic orientation frequently means orientation toward correct answers and toward the teacher's authority. It does not necessarily mean attention to and cultivation of children's interaction with each other, and of their ideas in general.

In addition, we believe the teacher deserves substantial credit for the students having this expertise. As previously noted, she nurtured these skills not only in this physics class, but also in her mathematics classes, in which these students also participated. Ms. K's classes were, from our observations, extraordinary in their attention to student ideas and to students' attention to each other's ideas, even (perhaps, especially) at this school. Several indicators of her work in this regard are her insistence on students explaining to each other nearly everything they do, her habit of having students recount previous work to absent students, her refusal to provide "the right answer" and her frequent reminder to address their remarks to other students, not to her.

***Interest.*** Student interest was a vital link in the success of this discussion. Without attention, learning is difficult to imagine. And without interest, rapt attention seems logically impossible. One needn't subordinate interest as merely instrumental to conceptual or practical accomplishment. Creating a lively class and allowing students to experience science in such a context can well serve as independent goals, though goals best served together with a learning orientation.

The interest of these students was evident in many ways. One could see their enthusiasm on many occasions directly in the tenor of discussion. Students often seemed to come to class with ideas prepared, even though no homework was ever assigned. One student reported, after a day of absence, that he had tried to convince his caretaker to allow him to come just for the Boxer class; but she had refused. As in the case of the dispute over instantaneously stopped motion, the discussions sometimes left the classroom with the students.

We are not sure how to characterize their interest. Several features, however, seem important. The first is ownership: Essentially all the ideas discussed belonged to the students. Second, competence: It is far easier to be interested in something that one is competent in engaging. Thus, interest relies on the fact that they had meta-representational skills to invent and critique in the first place. Indeed, that so many ideas were introduced must have kept many students interested. There was no student who "owned" all the ideas, so the novelty of others' contributions may have counted significantly. The fast pace of the conversations was carried by the students themselves, though we don't think it hurt that the teacher slowed the discussion on occasion to make sure it was comprehensible and to solidify gains. It also seems reasonable to conjecture that the focus on a richly describable and intricate product contributed to holding attention. We will say more about this product orientation later.

Characterizing the subject matter as intrinsically interesting, however, is problematic. Many times they were very interested in motion, many times they were less so. If there was anything special about the subject of this encounter, it may have been specifically that it involved depicting and communicating about motion, engaging an existing competence that still had discernable progress to make.

The group nature of the activity may have been a factor. Many of these students clearly liked to "perform" at the blackboard. That we were videotaping could well have added to this. It was, in fact, the first group-videotaped activity in the course, so the novelty might have contributed.

With regard to the density and diversity of ideas, it seemed the size of the group was nearly optimal for the kind of discussion that evolved. Contributions from nearly all the students were absorbed into the discussion. And there was a sufficient order and frequent enough "air time" for individual students to keep them engaged. How much bigger or smaller the group could have been is hard to say, but it is likely there would be difficulties either way.

***The Teacher's Contributions.*** In most classrooms, the teacher is responsible for providing and judging essentially all of the conceptual content of the class. Ms. K did very little in this regard in these discussions. She set the task, provided some initial criteria ("as simple as possible"), nudged a bit in judgment one way or another (a strong move: "Numbers are getting very precise; I like that, but can you show amount without numbers?"). It happened that she suppressed some student representations. Telephone poles, for example, were not among the sanctioned representations for the game on Day 4, even though one student declared it her "favorite." But, by and large, she stayed overtly out of content matters. Instead, she played a much more substantial but delicately balanced role in pragmatic affairs. She organized and facilitated the interaction. She provided conceptual focus indirectly by initiating and leading activities like the "words of motion" activity and the game of portraying selected motions in the various representations. She also provided conceptual focus by asking questions: "Whose shows stop?" "Why do you think Ts are better than chalk?" She took a major role in having the class revisit problematic issues, for example, she reprised the "time or distance on the horizontal" issue. In general, she took pragmatic initiatives to maintain proper articulation of activities and conceptual matters.

Although the classroom appeared sometimes confused to the point of near chaos (as one should expect from a student-oriented class), Ms. K made relatively frequent and sometimes very strong organizational moves. Some of these were relatively local, managing interactions of the group or the individuals in it. Most were much more global, keeping track of long-term goals (which she chose carefully with respect to conceptual encounters), for example, reminding the students that the task had two parts, depicting and explaining the depiction. She stopped some discussions to have students do individual seat work. When things got out of hand interactionally, she negotiated terms for continuation. She called for consensus on certain occasions, but she made productive use of many divergences as well. She set goals but would let students derail them if a different direction appeared more productive. For example, the task of naming the representations was deferred in favor of a discussion about whether to represent distance or speed in Ts. Ms. K eventually brought the class back to the naming task, but she totally abandoned her plan for a discussion about graphing backwards motion (negative speed) to allow the debate about the stopped state.

Ms. K's skill in deciding which of the students' directions to pursue is too delicate to examine in detail here. But we believe it was critical that she exercised such judgement often. Certainly, she did not take all suggestions. The students declared that they would like the game on Day 4 better if it were competitive and if they could make up their own motions to represent, but Ms. K insisted on her motions and had a point to prove that noncompetitive games can be also be fun.

One of Ms. K's best moves was to keep alive multiple, child-originated representations. She *systematized* them (for example she returned her own drawn versions of them to the students and asked that they decide whether distance or time was to define the horizontal dimensions), *named* them, and *exercised* them. Having the students' investment certainly is one value of this strategy, but it also played a central role in what happened conceptually. Note, by way of example, that graphing seemed to emerge from ideas the students devised cumulatively, with Ms. K's help in accumulating. In addition, the advantages and disadvantages of graphing emerged from the task of comparing it to other representations. Both the goals and means of adequate representation emerged in the engineering task of designing graphing, articulated in the failures and successes of various representations, for which Ms. K constantly probed.

### 6.4 Instructional View

Finally, we wish to characterize inventing graphing in ways that suggest useful and, we hope, generalizable instructional strategies. First, we will characterize inventing graphing as an activity built on a surprising child competence. This has two parts: choice of subject (meta-representation) and choice of mode (group design and invention). Then we will view inventing graphing as an instantiation of one of the central heuristics of a child's science of motion: viewing the child as builder. Finally, we will look at inventing graphing as instantiating a view of learning intellectual skills as tool appropriation, which happens best in functional contexts and with explicit concern for the principles by which tools function.

***Child Competence.*** From what we have said, it should be plain that we believe designing to engage competence is a primary instructional strategy. To the extent that consideration of meta-representational skills is new, this work establishes new instructional foci.

The existence of this pool of knowledge may well be important in sorting out certain difficulties in learning. For example, it seems within the realm of possibility that good students learn how to graph and use other specific representational techniques because they can essentially reinvent them on their own with the limited help that ordinary instruction provides. Thus, the appropriate locus of instruction for children who just do not "get it" may not be in any representation-specific skills, but at the more general level of meta-representation.

Although it is possible to focus on meta-representation in modes other than through design and invention, it seems to us these are very apt activity structures for engaging this kind of knowledge. Can children invent, in some reasonable sense, algebra or decimal notation? We think the answer to this is yes, but clearly this requires further research. At least we have found that inventing graphing is much more possible than one might expect.

Most technologically oriented instructional innovation involves providing students with a representation or with tools for understanding conventional representations. To be sure, much of our own course proceeds in this way. Often these approaches are billed as constructivist in that they involve the students in constructing meaning for themselves of the given representation. And, as such, they are often quite successful. However, we note how rare it is to find instruction that trusts children to create their own representations. Traditional representations are usually treated as sacred, close to the core of what we wish to teach. How can we negotiate them with children, or worse, trust children to invent them? Surely children cannot reproduce in short order what took civilization thousands of years to build. If they could, why should we bother to teach at all?

In asking students to design representations, we are experimenting with a deeper constructivism. We claim this has several potential advantages. For one, it closes the gap between students' prior knowledge and the material they are involved with (continuity of ideas). And it provides an opportunity for creative engagement and ownership of conceptually difficult material (continuity of activities). Last, it has students exercise meta-representational knowledge, which we expect to be of value for understanding any new representation. One of the difficulties with conventional instruction, we believe, is that students' metaknowledge is often not engaged, and so they may come to know "how to graph" without understanding what graphs are for or why the conventions make sense.

We've tried to begin building the case here that particular representations may not be at the core of what we should teach so much as the uses they serve, criteria they meet, and resources they build on. We've tried to underline that children already have a lot that prepares them for graphing, so that some of the final moves may be safely retraced by them.[9] We believe the extent to which knowledge is determined by activity is greatly underestimated; so, setting the task to be representing *particular* aspects of motion is quite a good hint. And shouldn't starting with what is to be represented help clarify what representations are about? Surely the meta-representational skills so extravagantly evident in some of this work cannot be absent in "more ordinary" children. Or, again, perhaps these are precisely the skills we need to build, not highly tuned graphing skills. Perhaps these are what separate the kids who get it from those who don't.

***Child as Builder.*** The second instructional view starts with the root metaphor of a child's science of motion, the child as engineer, as maker, rather than child as scientist. The basic claim, in this primitive form, is that children like to make things, that making things is a class of activity that more legitimately lies within the span of authentic child action, and that making things has natural and substantial genetic paths toward scientific activity. We meant explicitly to reject doing experiments and formulating laws as our central focus.[10] Although we started with a focus on children making things like programs and games, inventing graphing lets us see another side of the engineering metaphor. We should extend it to making things of a rather different sort: the child as designer of representational forms. Designing these can still involve ownership, visible products, community contributions, and use. And, we argue, designing representational forms is every bit as much at the core of science as experiments.

***Tools.*** The third and final instructional perspective is that learning intellectual skills can profitably be seen as appropriating tools. Fundamentally, tools need to

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[9] See Strauss and Schneider's (in press) work on young children's early and preinstruction concepts of graphing.

[10] Although, on occasions, we had students both do experiments and formulate physical laws. The meaning of our engineering frame is not in individual activities, but in the basic orientation toward engaging children's constructive skills and propensities.

be understood in the contexts in which they are useful: One understands a tool first by what it accomplishes. A focus on the isolated skills that constitute tool use is fragile and usually ineffective because it lacks the important feedback of function, which generates tools in the first place. Also, understanding a tool requires understanding the principles by which it works, or else the relation of the tool to its use is invisible. Thus, one may best know when a tool is appropriate, inappropriate, or must be modified, or when and how related tools can be invented, by understanding the principles by which the tool works. In particular, we have advocated that children may often learn tools best by inventing them out of a sense for what is to be accomplished and a sense for how the tool is to work. Even if they cannot design the fully tuned professional tool, function and principles can have their effects by helping to assure robust and flexible understanding.[11]

We believe that a focus on function and principles, particularly through design, characterizes very well what happened with inventing graphing. We believe that, as a result, these children are in a much better position to understand graphing and similar intellectual tools. For having engaged in the process of design, they are better prepared to invent replacements when graphs are inadequate.

Designers and engineers generate what they design and build. These children knew that they were responsible for the creation of respectable representations of motion. In general, then, design and engineering make a good starting place for good conceptions of the task of doing science itself, for we think it is important that children understand that science is not simply discovered and validated. In a real and important sense, but one that is not inherent in most scientific activities presented in schools, science is constructed. We hope doing science through design and engineering activities can foster a feeling of responsibility and competence in inventing science. We hope that such experiences as inventing graphing teaches children that they can, and indeed, *must* (re)invent even those things that are *presented to them* in the classroom.

## 7. CONCLUDING NOTE

We return for a final word on what must be a fundamental question about a case study such as this. Can inventing graphing ever happen again? There is no doubt that many features that led to this series of discussions were chance or apparently uncontrollable. Some students made extraordinary contributions that one could not count on. The student who proposed grids did so at nearly an ideal time.

First, we believe that the wonderful accidents were not so accidental. Triangles, after all, were one of the very first inventions, and differed substantially

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[11] This concept of instruction as tool appropriation is developed in diSessa (1990).

from the final product only in using distance instead of time on the horizontal axis. So, the fact that it did not take hold may have been in the dynamic, and not accidental. The idea of making a grid took hold in the group when many of the precursor ideas, use of an independent dimension, the value of continuity, and so on, had been established in the group as a whole. Consensus on graphing as the best representational form built only gradually.

Second, we have said that the very idea of meta-representation as a learning focus may be one important dimension on which this event may be generalized instructionally. We see, in the skills and concepts that these children exhibited, resources for teaching in whatever form. For example, a teacher might present some of these alternate representational forms, asking students to compare what they are able to depict or how easy they are to read.

Third, we are committed to being prepared for rare events. We do not believe that singular occurrences must be designed out of schools because they are so hard to reproduce. Minimally (but not trivially), this occurrence can serve to help us prepare for other wonderful and perhaps unplannable things that can happen. Perhaps we can get better at making them happen more regularly.[12]

Finally, we continue to advocate the principle high-level goals for a child's science of motion. Inventing graphing has been a good context in which to elaborate them, if not to validate them. Perhaps more than anything else, we advocate finding out when and how we can trust our students, so they can learn to take responsibility for the construction of scientific ideas.

## REFERENCES

Bamberger, J. (in press). *The mind behind the musical ear.* Cambridge, MA: Harvard University Press.

Brown, A. L., & Ferrara, R. A. (1985). Diagnosing zones of proximal development. In J. V. Wertsch (Ed.), *Culture, communication and cognition: Vygotskian perspectives.* Cambridge: Cambridge University Press.

Clement, J. (1982). Students' preconceptions in introductory mechanics. *American Journal of Physics, 50,* 66–71.

Clement, J. (1989). The concept of variation and misconceptions in Cartesian graphing. *Focus on Learning Problems in Mathematics, 11,* 77–87.

Cole, M. (1985). The zone of proximal development: Where culture and cognition create each other. In J. V. Wertsch (Ed.), *Culture, communication and cognition: Vygotskian perspectives.* Cambridge: Cambridge University Press.

diSessa, Andrea (1982). Unlearning Aristotelian physics: A study of knowledge-based learning. *Cognitive Science, 6,* 37–75.

diSessa, Andrea (1986). Artificial worlds and real experience. *Instructional Science 14,* 207–227. (Reprinted in R. Lawler and M. Yazdani (Eds.), *Artificial intelligence and education.* Norwood, NJ: Ablex, 1987.)

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[12] See Duckworth (1987) for a delightful discussion of this issue. See also diSessa (1990) on designing for rare experiences.

diSessa, Andrea A. (1990). Social niches for future software. In M. Gardner, J. Greeno, F. Reif, A. Schoenfeld, A. diSessa, & E. Stage (Eds.), *Toward a scientific practice of science education,* Hillsdale, NJ: Erlbaum.

Duckworth, Eleanor (1987). *"The having of wonderful ideas" and other essays on teaching and learning.* New York: Teachers College Press.

Kliman, M. (1987). Children's learning about the balance scale. *Instructional Science, 15,* 307–340.

Leinhardt, G., Zaslavsky, O., & Stein, M. K. (1990). Functions, graphs, and graphing: Tasks, learning and teaching. *Review of Educational Research, 60,* 1–64.

McCloskey, M., Caramazza, A., & Green, B. (1980). Curvilinear motion in the absence of external forces: Naive beliefs about the motion of objects. *Science, 210,* 1139–1141.

McDermott, L.C. (1984). Research on conceptual understanding in mechanics. *Physics Today, 37,* 24–32.

McDermott, L.C., Rosenquist, M.L., & van Zee, E.H. (1987). Student difficulties in connecting graphs and physics: Examples from kinematics. *American Journal of Physics, 55,* 503–513.

Minstrell, J. (1989). Teaching science for understanding. In L. Resnick & L.E. Klopfer (Eds.), *Toward the thinking curriculum: current cognitive research* (1989 Yearbook of the Association for Supervision and Curriculum Development). Alexandria, VA: ASCO

Strauss, S., & Schneider, A. (in press). The development of young children's comprehension and production of graphs. In S. Strauss (Ed.), *Development and learning environments.* Norwood, NJ: Ablex.

Trowbridge, D.E., & McDermott, L.C. (1980). Investigation of student understanding of the concept of velocity in one dimension. *American Journal of Physics, 48,* 1020–1028.

Trowbridge, D.E., & McDermott, L.C. (1981). Investigation of student understanding of the concept of acceleration in one dimension. *American Journal of Physics, 49,* 242–253.

van Zee, E.H., & McDermott, L.C. (1987). Investigation of student difficulties with graphical representations in physics. In J. Novak (Ed.), *Proceedings of the second international seminar on misconceptions and educational strategies in science and mathematics.* Ithaca, NY: Cornell University Press.

Viennot, L. (1979). Spontaneous reasoning in elementary dynamics. *European Journal of Science Education, 1,* 205–221.

Wertsch, J.V. (1985). *Culture, communication and cognition: Vygotskian perspectives.* Cambridge: Cambridge University Press.

Wertsch, J. V., & Stone, C. A. (1985). The concept of internalization in Vygotsky's account of higher mental functions. In J. V. Wertsch (Ed.), *Culture, communication and cognition: Vygotskian perspectives.* Cambridge: Cambridge University Press.